1. (15 points) Let $A, B \in C^{m\times m}$ be arbitrary matrices. Show that

$$\|AB\|_F \leq \|A\|_2 \|B\|_F,$$

where $\|\cdot\|_2$ and $\|\cdot\|_F$ denote the 2-norm and Frobenius norm, respectively.

2. (15 points) Fix $0 < \varepsilon < 1$ and suppose that $A \in R^{m \times m}$ is symmetric and nonsingular. Show that if $\|A - I\|_F \geq \varepsilon$, then $\|A^{-1} - I\|_F \geq \frac{\varepsilon}{2}$, where $\| \cdot \|_F$ denotes the Frobenius norm.

3. (10 points) Prove that the determinant of a Householder reflector is negative one.

4. (15 points) Let $\varepsilon > 0$ be given, $k << \min(m,n)$, $A \in R^{m\times n}$, $C \in R^{m\times k}$, and $B \in R^{k\times n}$. Assume that

$$\|A - CB\| \leq \epsilon,$$

where $\|\cdot\|$ denotes the matrix 2-norm, and B and C have rank k. Further suppose that A is not available, and only B and C are available. **Without forming** the product of C and B, design an efficient algorithm to compute an approximate reduced QR of A so that the following holds,

$$\|A - QR\| \leq \varepsilon,$$

where Q is an orthonormal matrix and R is upper triangular.

5. (15 points) Show that if $A \in \mathcal{R}^{n\times n}$ is symmetric, then for $k = 1$ to n,

$$\lambda_k(A) = \max_{\dim(S)=k} \min_{\mathbf{0}\neq\mathbf{y}\in S} \frac{\mathbf{y}^T A\mathbf{y}}{\mathbf{y}^T\mathbf{y}},$$

where S is a subspace of $\mathcal{R}^n$, and $\lambda_k(A)$ designates the kth largest eigenvalue of A so that these eigenvalues are ordered,

$$\lambda_n(A) \le \cdots \le \lambda_2(A) \le \lambda_1(A).$$

6. Let $A \in \mathcal{R}^{m\times n}$, $\text{rank}(A) = r$, and $\mathbf{b} \in \mathcal{R}^m$, and consider the system $A\mathbf{x} = \mathbf{b}$ with unknown $\mathbf{x} \in \mathcal{R}^n$. Making no assumption about the relative sizes of n and m, we formulate the following least-squares problem:

   of all the $\boldsymbol{x} \in \mathcal{R}^n$ that minimizes $\|\boldsymbol{b} - A\boldsymbol{x}\|_2$, find the one for which $\|\boldsymbol{x}\|_2$ is minimized.

   (a) (5 points) Show that the set Γ of all minimizers of the least-squares function is a closed convex set:

   $$\Gamma = \{\mathbf{x} \in \mathcal{R}^n : \|A\mathbf{x} - \mathbf{b}\|_2 = \min_{\mathbf{v}\in\mathcal{R}^n} \|A\mathbf{v} - \mathbf{b}\|_2\}.$$

   (b) (5 points) Show that the minimum-norm element in Γ is unique.

   (c) (5 points) Show that the minimum norm solution is $\mathbf{x} = A^+\mathbf{b} = V\Sigma^+U^*\mathbf{b}$, where $A = U\Sigma V^*$, and Σ^+ is the pseudo-inverse of Σ.

7. Consider the following linear system,

$$A\mathbf{x} = F, \tag{1}$$

where

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & -1 & 2 & -1 \\ \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}$$

(a) (5 points) Prove that the $n \times n$ tridiagonal matrix A is symmetric, positive definite (SPD).

(b) (5 points) Let B be a tridiagonal SPD matrix in the form of the matrix A. Prove that the Cholesky factor L of B has nonzero entries only along the main diagonal and the sub-diagonal lines, where $B = LL^t$. Give the formula for L.

(c) (5 points) Design an $O(n)$ algorithm to solve the linear system $A\mathbf{x} = F$.